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On Systems of Multiform Functions belonging to a Group of Linear Substitutions with Uniform Coefficients.

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INTRODUCTION.

The integrals of a linear differential equation with uniform coefficients have the characteristic property of being uniform and continuous everywhere except in the vicinity of the singular points of the equation, where they undergo, in general, linear substitutions with constant coefficients.

While Fuchs takes the differential equation to be given, and the essential problem for him is the determination of the substitution group belonging to its integrals, Riemann takes up the converse problem. He supposes the branch-points and the fundamental substitutions to be given arbitrarily. The question then arises as to the existence of a system of functions having the given substitutions and branch-points. In the cases treated by him, Riemann proves the existence theorem by passing to the differential equation and finding the arithmetical expressions for the functions. A direct proof, similar to that furnished by Schwarz and Neumann as basis for Riemann's theory of Abelian Functions, has been given by Klein for a very special case only.*

It will therefore not be surprising that we encounter still greater difficulties if we attempt to prove the existence of the general functions, studied in this paper, as I believe, for the first time. Nothing like a general and direct existence theorem is therefore to be found in it. But the existence of a large and important class of these functions is demonstrated by an indirect method, which consists essentially in generalizing the hypergeometric functions in a proper manner.

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* Klein in Ritter's memoir, Math. Ann., Bd. 41.

The first part of the paper treats of the theory of the functions, so far as concerns their behavior in the vicinity of any singular point. This suffices to establish the existence theorem in some very simple special cases. The latter part of the paper deals with the generalized hypergeometric functions.

§1.—*Formulation of the Problem.*

Let $(y_1, y_2, \ldots, y_n)$ be a system of n functions of x, which are uniform everywhere except in the vicinity of m points, called branch-points, $a_1, a_2, \ldots, a_m$. Suppose moreover that when x makes a circuit around a_i in the positive direction, the system $(y_1, y_2, \ldots, y_n)$ undergoes a linear substitution with *variable* coefficients. It is therefore supposed that the coefficients of the substitutions are themselves functions of x, and, as we shall mostly assume, *uniform* functions. But we shall also to some extent consider the case that these coefficients are multiform with a finite number of branches.

We will assume that the n functions $(y_1, \ldots, y_n)$ are *independent*, signifying thereby that no relation of the form

$$\psi_1 y_1 + \psi_2 y_2 + \ldots + \psi_n y_n = 0$$

can be identically verified, where $\psi_1, \psi_2, \ldots, \psi_n$ are uniform functions of x in the vicinity of the singular points a_i.

Let

$$A_i = \begin{pmatrix} \alpha_{11}^{(i)}, & \alpha_{12}^{(i)}, & \ldots, & \alpha_{1n}^{(i)} \\ \alpha_{21}^{(i)}, & \alpha_{22}^{(i)}, & \ldots, & \alpha_{2n}^{(i)} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1}^{(i)}, & \alpha_{n2}^{(i)}, & \ldots, & \alpha_{nn}^{(i)} \end{pmatrix}, \quad (i = 1, 2, \ldots, m) \tag{1}$$

be the substitution which $(y_1, \ldots, y_n)$ undergoes when x describes a positive circuit around a_i.

Obviously the relation must hold

$$A_1 A_2 \ldots A_m = 1 \tag{2}$$

if 1 denotes identity. This suffices to show that no such systems of functions, which for briefness we will call Λ functions, are possible with only one branch-point. If there are two branch-points the two substitutions are inverses of each other.

It is the purpose of this paper to develop the general properties of the Λ

functions, and to establish their existence in some general cases. The analogy to the theory of linear differential equations is striking, and most of our results will be found by methods familiar to those who are interested in that theory.

§2.—*The Characteristic Equation.*

Let a be any one of the branch-points, and A the corresponding substitution. If we put

$$z = \lambda_1 y_1 + \lambda_2 y_2 + \ldots\ldots + \lambda_n y_n, \tag{3}$$

$\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ can generally be so determined as functions of x, uniform in the vicinity of $x = a$, that after a circuit around $x = a$, *z will change into $\bar{z} = \omega z$, where ω is also uniform in the vicinity of $x = a$.*

In order that this may be so, the equations

$$\left.\begin{array}{l} \lambda_1(\alpha_{11} - \omega) + \lambda_2\alpha_{21} + \ldots\ldots + \lambda_n\alpha_{n1} \qquad = 0, \\ \lambda_1\ \alpha_{12} + \lambda_2(\alpha_{22} - \omega + \ldots\ldots + \lambda_n\alpha_{n2} \qquad = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \lambda_1\ \alpha_{1n} + \lambda_2\ \alpha_{2n} \quad + \ldots\ldots + \lambda_n(\alpha_{nn} - \omega) = 0 \end{array}\right\} \tag{4}$$

must be verified, since $y_1, \ldots\ldots, y_n$ are supposed to be independent. Therefore, the case $\lambda_1 = \lambda_2 = \ldots\ldots = \lambda_n = 0$ being of course excluded, we must have

$$F(\omega) = \begin{vmatrix} \alpha_{11} - \omega, & \alpha_{21} & , \ldots\ldots, \alpha_{n1} \\ \alpha_{12} & , \alpha_{22} - \omega, & \ldots\ldots, \alpha_{n2} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots \\ \alpha_{1n} & , \alpha_{2n} & , \ldots\ldots, \alpha_{nn} - \omega \end{vmatrix} = 0, \tag{5}$$

which is the *characteristic equation.* The above theorem will then, as equations (3), (4), (5) show, be true if $x = a$ is not a branch-point of the characteristic equation.

In general this equation will have n unequal roots, ω_i. *If $x = a$ is no branch-point for any of them, there will be n linear homogeneous combinations of $(y_1, \ldots\ldots, y_n)$, which we will call $(z_1, \ldots\ldots, z_n)$ such that*

$$\bar{z}_i = \omega_i z_i, \qquad (i = 1, 2, \ldots\ldots, n) \tag{6}$$

denoting always by a dash over a quantity the value which it assumes after a positive circuit around $x = a$.

The substitution expressed by (6) is said to be in the *canonical* form, and the system $(z_1, \ldots\ldots, z_n)$ is to be called the canonical system belonging to $x = a$.

It is important to distinguish whether $F(\omega)$ is reducible or not. If it is irreducible the n roots $\omega_1, \ldots, \omega_n$ corresponding to each value of x are but the n branches of a single monogenic function of x, so that any one root ω_i can be analytically continued into every other root ω_k, or, as we may also say, from every point of the Riemann's surface belonging to (5) there is a path to every other point of that surface. The multipliers ω_i will then be permutated if x describes closed circuits in that Riemann's surface. If $F(\omega) = 0$ is composed of several irreducible factors, only the roots belonging to each factor are thus interchangeable. In particular, it may happen that $F(\omega)$ is a product of n linear factors. Then $\omega_1, \ldots, \omega_n$ are absolutely uniform functions of x.

Let us now assume that $x = a$ is a branch-point of the characteristic equation, where the μ roots $\omega_1, \omega_2, \ldots, \omega_\mu$ are interchanged cyclically, so that

$$\omega_1 = \omega_2, \quad \bar{\omega}_2 = \omega_3, \quad \ldots, \quad \bar{\omega}_{\mu-1} = \omega_\mu, \quad \bar{\omega}_\mu = \omega_1.$$

Let us moreover denote the λ's which result from (4), when ω is put equal to ω_i, by λ_{ik} $(k = 1, 2, \ldots, n)$ and the corresponding z resulting from (3) by z_i. *Then obviously if we determine* $z_1, \ldots, z_\mu$ *by the same equations* (3), (4), (5) *which hold in the general case,*

$$\bar{z}_1 = \omega_2 z_2, \quad \bar{z}_2 = \omega_3 z_3, \quad \ldots, \quad \bar{z}_{\mu-1} = \omega_\mu z_\mu, \quad \bar{z}_\mu = \omega_1 z_1, \tag{7}$$

and similarly for any group of roots which has $x = a$ as a branch-point. These equations will in this case be termed the canonical substitutions instead of (6).

If $x = a$ *is no branch-point and* $\omega_1, \ldots, \omega_n$ *are unequal roots of* $F(\omega) = 0$, $z_1, \ldots, z_n$ *are independent.* For if there were a relation with coefficients γ_i uniform for $x = a$,

$$\gamma_1 z_1 + \gamma_2 z_2 + \ldots + \gamma_n z_n = 0,$$

by making circuits around $x = a$, we would find

$$\begin{aligned} \omega_1 \gamma_1 z_1 \quad &+ \omega_2 \gamma_2 z_2 \quad + \ldots + \omega_n \gamma_n z_n \quad = 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \omega_1^{n-1} \gamma_1 z_1 &+ \omega_2^{n-1} \nu_2 z_2 + \ldots + \omega_n^{n-1} \gamma_n z_n = 0. \end{aligned}$$

But unless $\gamma_1 = \gamma_2 = \ldots\ldots = \gamma_n = 0$, this is impossible, because the determinant

$$\begin{vmatrix} 1 & , & 1 & , \ldots\ldots , & 1 \\ \omega_1 & , & \omega_2 & , \ldots\ldots , & \omega_n \\ \cdots & & \cdots & \cdots & \cdots \\ \omega_1^{n-1}, & & \omega_2^{n-1}, & \ldots\ldots , & \omega_n^{n-1} \end{vmatrix}$$

$$= (\omega_1 - \omega_2) \ldots\ldots (\omega_1 - \omega_n)(\omega_2 - \omega_3) \ldots\ldots (\omega_2 - \omega_n) \ldots\ldots (\omega_{n-1} - \omega_n)$$

cannot vanish for unequal values of $\omega_1, \ldots\ldots, \omega_n$.

Now let $x = a$ be a branch-point. Obviously after μ circuits around a, $(z_1, \ldots\ldots, z_\mu)$, the canonic functions belonging to the cycle of roots $\omega_1, \ldots\ldots, \omega_\mu$ will have changed into $\omega_1^\mu z_1, \ldots\ldots, \omega_\mu^\mu z_\mu$ exactly as if $\omega_1, \ldots\ldots, \omega_\mu$ were uniform for $x = a$. Suppose that there are still other groups of $\nu, \rho, \ldots\ldots$ roots, the members of each group being permutated when x describes a circuit around $x = a$. Let σ be the least common multiple of μ, ν, ρ, etc. Then from a relation

$$\gamma_1 z_1 + \gamma_2 z_2 + \ldots\ldots + \gamma_n z_n = 0$$

would follow by making $\sigma, 2\sigma, \ldots\ldots, (n-1)\sigma$ circuits

$$\begin{aligned} \omega_1^\sigma \gamma_1 z_1 \quad &+ \omega_2^\sigma \gamma_2 z_2 \quad + \ldots\ldots + \omega_n^\sigma \gamma_n z_n \quad = 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \omega_1^{(n-1)\sigma} \gamma_1 z_1 &+ \omega_2^{(n-1)\sigma} \gamma_2 z_2 + \ldots\ldots + \omega_n^{(n-1)\sigma} \gamma_n z_n = 0. \end{aligned}$$

The determinant of these n equations

$$(\omega_1^\sigma - \omega_2^\sigma) \ldots\ldots (\omega_1^\sigma - \omega_n^\sigma)(\omega_2^\sigma - \omega_3^\sigma) \ldots\ldots (\omega_2^\sigma - \omega_n^\sigma) \ldots\ldots (\omega_{n-1}^\sigma - \omega_n^\sigma)$$

can only vanish if the quotient of two ω's is a σ^{th} root of unity. *Therefore if $x = a$ is a branch-point of the characteristic equation, and if the quotient of no two roots of this equation is a root of unity, the system of canonical junctions, which we have defined, consists of n independent members.*

The cases of equal roots, or if $x = a$ is a branch-point of roots whose quotients are roots of unity, will be treated hereafter.

§3.—*Invariance of the Characteristic Equation.*

Before we can treat the cases mentioned at the close of the last paragraph, we must notice the following theorem:

If instead of $(y_1, y_2, \ldots\ldots, y_n)$ we consider the system of functions

$$\eta_i = \mu_{i1} y_1 + \mu_{i2} y_2 + \ldots\ldots + \mu_{in} y_n, \quad \text{Det}\,(\mu_{ik}) \neq 0,$$
$$(i = 1, 2, \ldots\ldots, n)$$

where μ_{ik} *are functions of* x *which are uniform in the vicinity of* $x = a$, *then the characteristic equation belonging to the* η *functions is the same as that for* $(y_1, y_2, \ldots, y_n)$.

To prove this we can proceed exactly as in the theory of linear differential equations. It is therefore unnecessary to reproduce the demonstration.

§4.—*Form of the Canonical Substitutions when the Characteristic Equation has Equal Roots, or Roots whose Quotient is a Root of Unity.*

Let $x = a$ be no branch-point of the λ roots of $\omega_1, \omega_2, \ldots, \omega_\lambda$, which we will suppose to be equal. Then there will be one function z_1 for which

$$\bar{z}_1 = \omega_1 z_1 .$$

Since $y_1, \ldots, y_n$ are supposed to be independent, $z_1, y_2, \ldots, y_n$ will be likewise. Then we will have

$$\begin{cases} \bar{z}_1 = \omega_1 z_1 , \\ \bar{y}_2 = \beta_{21} z_1 + \beta_{22} y_2 + \ldots + \beta_{2n} y_n , \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \bar{y}_n = \beta_{n1} z_1 + \beta_{n2} y_2 + \ldots + \beta_{nn} y_n , \end{cases}$$

where β_{ik} and ω_1 are uniform in the vicinity of $x = a$. The corresponding characteristic equation becomes

$$\begin{vmatrix} \omega_1 - \omega, & \beta_{21} & , \ldots, & \beta_{n1} \\ 0 \ , & \beta_{22} - \omega, & \ldots, & \beta_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ 0 \ , & \beta_{2n} & , \ldots, & \beta_{nn} - \omega \end{vmatrix} = 0,$$

so that, since $F(\omega) = 0$ has the root ω_1 λ times, according to the theorem of the invariance of the characteristic equation, the equation

$$\begin{vmatrix} \beta_{22} - \omega, & \ldots, & \beta_{n2} \\ \ldots & \ldots & \ldots \\ \beta_{2n} & , \ldots, & \beta_{nn} - \omega \end{vmatrix} = 0,$$

has this root $\lambda - 1$ times. The following system of linear equations

$$\beta_{2i} x_2 + \beta_{3i} x_3 + \ldots + \beta_{ni} x_n = \omega_1 x_i$$
$$(i = 2, 3, \ldots, n)$$

therefore admits of solution, and if we put

$$z_2 = x_2 y_2 + x_3 y_3 + \ldots + x_n y_n ,$$

we shall find

$$\begin{aligned}\bar{z}_2 = z_1(\beta_{21}x_2 + \beta_{31}x_3 + \dots + \beta_{n1}x_n)\\ + \sum_{i=2}^{n} y_i(\beta_{2i}x_2 + \beta_{3i}x_3 + \dots + \beta_{ni}x_n).\end{aligned}$$

or

$$\bar{z} = \omega_{21}z_1 + \omega_1 z_2,$$

where ω_{21} is also uniform in the vicinity of $x = a$. If ω_1 is absolutely uniform so is ω_{12}.

Now $z_1, z_2, y_3, \dots, y_n$ can be taken as independent members of a system, and we can continue this process so as to find the following theorem:

If ω_1 is a root of the characteristic equation of multiplicity λ, there exists a group of λ functions $z_1, z_2, \dots, z_\lambda$ which take the following values when x describes a positive circuit around $x = a$:

$$\left.\begin{array}{l}\bar{z}_1 = \omega_1 z_1,\\ \bar{z}_2 = \omega_{21}z_1 + \omega_1 z_2,\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ \bar{z}_\lambda = \omega_{\lambda 1}z_1 + \omega_{\lambda 2}z_2 + \dots + \omega_1 z_\lambda.\end{array}\right\} \tag{8}$$

If ω_1 is an absolutely uniform function of x, the same is true of the quantities ω_{ik}.

If $x = a$ is a branch-point for the μ interchanging roots $\omega_1, \dots, \omega_\mu$ of the characteristic equation, these are of course unequal. The case of equal roots can then only occur if the irreducible factor $f(\omega)$ of $F(\omega)$, of which ω_1 is a root, occurs more than once. Therefore $\omega_1, \omega_2, \dots, \omega_\mu$ will each occur the same number of times λ if $f(\omega)^\lambda$ is the highest power of $f(\omega)$, which is a factor of $F(\omega)$.

Now for one circuit around $x = a$

$$\bar{z}_1 = \omega_2 z_2, \dots, \bar{z}_{\mu-1} = \omega_\mu z_\mu, \bar{z}_\mu = \omega_1 z_1,$$

and for μ circuits

$$\bar{z}_1 = \omega_1^\mu z_1, \dots, \bar{z}_\mu = \omega_\mu^\mu z_\mu.$$

If we put

$$x - a = t^\mu$$

μ circuits of x around $x = a$ will correspond to one of t around $t = 0$ and $\omega_1, \dots, \omega_\mu$ will be uniform as functions of t in the vicinity of $t = 0$. *Let $\omega_i, \omega_i', \dots, \omega_i^{(\lambda-1)}$ for $i = 1, 2, \dots, \mu$ be the λ roots which are equal to ω_i, then, corresponding to each of these i groups of λ equal roots there are λ functions*

$z_{i1}, \ldots . z_{i\lambda}$ *which after one circuit around* $t=0$, *or* μ *circuits around* $x=a$, *change into*

$$\left.\begin{aligned} \bar{z}_{i1} &= \omega_i^{\mu} z_{i1}, \\ \bar{z}_{i2} &= \omega_{21}^{(i)} z_{i1} + \omega_i^{\mu} z_{i2}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \bar{z}_{i\lambda} &= \omega_{\lambda 1}^{(i)} z_{i1} + \omega_{\lambda 2}^{(i)} z_{i2} + \ldots . + \omega_i^{\mu} z_{i\lambda}, \end{aligned}\right\} \qquad (9)$$
$$(i = 1, 2, \ldots . , \mu)$$

where ω_i *and the quantities* $\omega_{kl}^{(i)}$ *are uniform functions of* $(x-a)^{\frac{1}{\mu}}$ *in the vicinity of* $x=a$.

Finally, we must treat the case that $x=a$ is a branch-point, and that $\omega_1, \omega_2, \ldots . , \omega_\lambda$ only differ by factors from ω_1, which are σ^{th} roots of unity (§2). σ was the least common multiple of μ, ν, etc., where these integers gave the number of branches in each cycle of ω's which were permutated by a circuit around $x=a$. Since

$$\omega_1^{\sigma} = \omega_2^{\sigma} = \ldots . = \omega_\lambda^{\sigma},$$

this reduces the problem to the case of equal roots for a multiple circuit, as solved by equations (9).

§5.—*Analytical Form of the Canonical Functions in the Vicinity of the Branch-point to which they belong.*

In all of the different cases mentioned, it is possible to find analytical expressions for the canonical system of functions which hold in the vicinity of $x=a$.

Let us first consider the case that all of the roots of the characteristic equation are unequal, and uniform in the vicinity of $x=a$. Let $f(x)$ be uniform in the vicinity of $x=a$, and put

$$z = e^{f(x)\log(x-a)}.$$

Then, making a positive circuit around $x=a$,

$$\bar{z} = e^{2\pi i f(x)} z.$$

Putting therefore

$$f(x) = \frac{1}{2\pi i}\log\omega, \quad z = e^{\frac{1}{2\pi i}\log\omega\log(x-a)} \qquad (10)$$

we find

$$\bar{z} = \omega z, \qquad (10a)$$

if $\log\omega$ is uniform for $x = a$, i. e. if $x - a$ is neither a zero nor an infinity of finite multiplicity of ω. If a is an essential singularity, the expression (10) will not in general verify (10a). It will however do so, as is at once seen, if in the vicinity of $x = a$, ω can be expressed in the form

$$\omega = e^{P(x-a) + P_1\left(\frac{1}{x-a}\right)},$$

for then $\log\omega$ is uniform.

If a is a zero or infinity of ω, expression (10) must be modified. Suppose that it is a zero of multiplicity μ, so that

$$\omega = (x-a)^{\mu}\omega',$$

where ω' is neither zero nor infinite for $x = a$. If we put

$$z_1 = e^{\frac{1}{2\pi i}\log\omega'\log(x-a)},$$

then, if z is the required function for which $\bar{z} = \omega z$,

$$\bar{z}_1 = \omega' z_1, \quad \left(\overline{\frac{z}{z_1}}\right) = (x-a)^{\mu}\frac{z}{z_1},$$

or putting $\eta = \frac{z}{z_1}$,

$$\eta = (x-a)^{\mu}\eta.$$

Now let

$$z_2 = e^{\frac{\mu}{4\pi i}[\log(x-a)]^2}.$$

Then

$$\bar{z} = e^{\frac{\mu}{4\pi i}[\log(x-a)^2 + 4\pi i\log(x-a) - 4\pi^2]} = z_2(x-a)^{\mu}e^{-\frac{\mu\pi}{i}}.$$

Therefore if z_3 denotes η/z_2,

$$\bar{z}_3 = e^{-\mu\pi i}z_3,$$

which shows we may put

$$z_3 = (x-a)^{-\frac{\mu}{2}}.$$

Therefore we find

$$z = z_1 z_2 z_3 = e^{\frac{1}{2\pi i}\log(x-a)[\log\omega' + \frac{\mu}{2}\log(x-a) - \mu\pi i]}. \tag{11}$$

If $x = a$ is an infinity of multiplicity ν, μ in this expression must be replaced by $-\nu$.

Now let $E(\omega)$ denote either (10) or (11), as the case demands. If the roots of the characteristic equation are all unequal,

$$\overline{\left[\frac{z_i}{E(\omega_i)}\right]} = \frac{z_i}{E(\omega_i)},$$

i. e. denoting this quotient by $\phi_i(x)$, $\phi_i(x)$ is uniform in the vicinity of $x = a$, and can therefore be developed in a series of integral powers of $x - a$ containing in general an infinite number of positive and negative exponents. *We will therefore have*

$$z_i = E(\omega_i)\,\phi_i(x). \qquad (i = 1, 2, \ldots, n) \quad (12)$$

We need not give the detailed proof of the following theorem, as it can be obtained in a manner which is familiar to those interested in linear differential equations.

If ω_1 *is a multiple root of the characteristic equation, uniform in the vicinity of* $x = a$, *occurring* λ *times, the canonical functions belonging to this group of equal roots, whose behavior is expressed by equations* (8), *can be represented in the form*

$$\left.\begin{aligned} z_1 &= E(\omega_1)\,\phi_{11}(x), \\ z_2 &= E(\omega_1)[\phi_{21}(x) + \phi_{22}(x)\log(x - a)], \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ z_\lambda &= E(\omega_1)[\phi_{\lambda 1}(x) + \phi_{\lambda 2}(x)\log(x-a) + \ldots + \phi_{\lambda\lambda}(x)\log(x-a)^{\lambda-1}], \end{aligned}\right\} \quad (13)$$

where all of the ϕ *functions are uniform for* $x = a$. *If* ω_1 *is absolutely uniform, all of the* ϕ *functions can be expressed in homogeneous linear functions of* $\phi_{11}, \phi_{21}, \ldots, \phi_{\lambda 1}$ *with absolutely uniform coefficients. The coefficients of the highest powers of the logarithm, in particular, differ from* ϕ_{11} *only by an absolutely uniform factor.*

Now let $x = a$ be a branch-point for the μ interchanging branches $\omega_1, \omega_2, \ldots, \omega_\mu$. The canonical functions in this case are those for which

$$\bar{z}_1 = \omega_2 z_2, \ \ldots, \ \bar{z}_{\mu-1} = \omega_\mu z_\mu, \quad \bar{z}_\mu = \omega_1 z_1,$$

if the quotient of no two ω's is a μ^{th} root of unity. The expressions

$$e^{\frac{1}{2\pi i}\log\omega_i \log(x-a)} \qquad (i = 1, 2, \ldots, \mu)$$

have the same property if $\omega_1, \ldots, \omega_\mu$ do not vanish or become infinite for $x = a$, and a slight modification of (11) gives expressions which have the same property if $\omega_1, \ldots, \omega_\mu$ are zero or infinite for $x = a$. It is only necessary to multiply (11) by $(x - a)^{\pm\lambda}$ if λ is the multiplicity of the zero or pole.

Again calling these functions $E(\omega_i)$ in either case, we have

$$\overline{\left[\frac{z_1}{E(\omega_1)}\right]} = \frac{z_2}{E(\omega_2)}, \quad \overline{\left[\frac{z_2}{E(\omega_2)}\right]} = \frac{z_3}{E(\omega_3)}, \ \ldots, \ \overline{\left[\frac{z_\mu}{E(\omega_\mu)}\right]} = \frac{z_1}{E(\omega_1)},$$

i. e. if we put

$$\frac{z_i}{E(\omega_i)} = \psi_i(x),$$

$\psi_1(x), \ldots, \psi_\mu(x)$ are permutated by a circuit around $x = a$. *They are therefore the roots of an equation of degree μ whose coefficients are uniform in the vicinity of $x = a$, and $z_1, \ldots, z_\mu$ are represented by*

$$z_i = E(\omega_i)\,\psi_i(x). \qquad (i = 1, 2, \ldots, \mu) \tag{14}$$

In the vicinity of a, $\psi_i(x)$ will be developed in a series of positive and negative powers of $(x - a)^{\frac{1}{\mu}}$. If

$$\psi_1(x) = P\left[(x - a)^{\frac{1}{\mu}}\right] + P_1\left[\frac{1}{(x - a)}^{\frac{1}{\mu}}\right],$$

then

$$\psi_i(x) = P\left[(x - a)^{\frac{1}{\mu}}\alpha^{i-1}\right] + P_1\left[\frac{1}{(x - a)}^{\frac{1}{\mu}}\alpha^{i-1}\right],$$

where α is a primitive μ^{th} root of unity, namely,

$$\alpha = e^{\frac{2\pi i}{\mu}}.$$

If $x = a$ is a branch-point for the root ω_1, and if this root, and of course also the other roots belonging to the same cycle $\omega_2, \ldots, \omega_\mu$, occurs more than once, say λ times, there will be λ canonical functions corresponding to the root ω_1,

$$\left.\begin{aligned}
z_{11} &= E(\omega_1)\,\psi_{11}(x), \\
z_{12} &= E(\omega_1)[\psi_{21}(x) + \psi_{22}(x)\log(x - a)], \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
z_{1\lambda} &= E(\omega_1)[\psi_{\lambda 1}(x) + \psi_{\lambda 2}(x)\log(x - a) + \ldots + \psi_{\lambda\lambda}(x)\log(x - a)^{\lambda - 1}],
\end{aligned}\right\} \tag{15}$$

where all of the ψ functions are uniform functions of $t = (x - a)^{\frac{1}{\mu}}$ in the vicinity of $x = a$. If ω_1 is an absolutely uniform function of $(x - a)^{\frac{1}{\mu}}$, all of the ψ functions can be expressed as homogeneous linear functions of $\psi_{11}(x), \ldots, \psi_{\lambda 1}(x)$ with coefficients which are absolutely uniform functions of $(x - a)^{\frac{1}{\mu}}$.

A similar system exists for the $\lambda(\mu - 1)$ roots equal to $\omega_2, \omega_3, \ldots, \omega_\mu$ respectively.

If finally $\omega_1, \omega_2, \ldots, \omega_\nu$ are roots of the characteristic equation whose quotients are roots of unity, there exists a system of ν canonical functions of the same form as (15).

§6.—*Cogredient* Λ *functions, and functions of the same kind and class.*

We will call two Λ functions $(y_1, \dots, y_n)$ and $(y'_1, \dots, y'_n)$ *cogredient* if they have the same branch-points and the same group of substitutions. This term, borrowed from the theory of invariants, was used in the theory of linear differential equations as I believe for the first time by Klein. It is a generalization of the notions of *class* and *kind* which are due to Riemann and Poincaré.

We will say that two Λ functions belong to the same *kind** if they have not only the same branch-points and substitution group, but also have those singular points in common in whose vicinity they are uniform, but in which they have no definite, finite or infinite value.†

If, finally, for two Λ functions of the same kind the poles also coincide, they are said to belong to the same class.

We will here be principally concerned with the most general of these conceptions, which is that of cogrediency.

If $(y_1, y_2, \dots, y_n)$, $(y'_1, y'_2, \dots, y'_n)$, ..., $(y_1^{(\lambda-1)}, y_2^{(\lambda-1)}, \dots, y_n^{(\lambda-1)})$ are λ systems of cogredient Λ functions, obviously the system

$$y_i^{(\lambda)} = u_0 y_i + u_1 y'_i + \dots + u_{\lambda-1} y^{(\lambda-1)}, \qquad (i = 1, 2, \dots, n) \tag{16}$$

where $u_0, u_1, \dots, u_{\lambda-1}$ are any uniform functions, is also cogredient with $y_1, \dots, y_n$.

If between $\lambda + 1$ systems of cogredient Λ functions there is no relation of the form (16), they are said to be *independent. Then the theorem holds that there can be no more than n independent cogredient systems of* Λ *functions.*

For if $(y_1, \dots, y_n)$, ..., $(y_1^{(n)}, \dots, y_n^{(n)})$ are $n+1$ systems of cogredient Λ functions, and we define $g_1, \dots, g_n$ by

$$y_i + g_1 y'_i + \dots + g_n y_i^{(n)} = 0, \qquad (i = 1, 2, \dots, n) \tag{17}$$

it follows at once that $g_1, \dots, g_n$ are uniform functions of x. For $g_1, \dots g_n$ are equal to the quotients of the determinants which are obtained from the matrix

$$\left\| \begin{matrix} y_1, \; y'_1, \dots, y_1^{(n)} \\ \dots\dots\dots\dots \\ y_n, \; y'_n, \dots, y_n^{(n)} \end{matrix} \right\|$$

---

* We have translated M. Poincaré's "espèce" by kind.

† Called by Fuchs "Unbestimmtheitsstellen."

by omitting always one vertical line. Now when x describes a circuit around a_i each determinant is multiplied by the determinant of the substitution A_i. Their quotients are therefore uniform.

This result is perfectly analogous to Riemann's theorem, and can be stated as follows:

Any $n+1$ *systems of cogredient* Λ *functions verify a homogeneous linear relation with uniform coefficients.*

The same is true of $n+1$ systems of the same kind or class, except that the uniform coefficients must then be subjected to certain conditions, as in the following theorem:

If the development of the canonical functions in the vicinity of every singular point, and for all of the $n+1$ systems, contains only a finite number of negative powers, the coefficients of the linear relation are rational functions.

§7.—*Linear Differential Equations.*

When the fundamental substitutions have constant coefficients, $\left(\frac{dy_1}{dx}, \frac{dy_2}{dx}, \ldots., \frac{dy_n}{dx}\right)$ are of the same class as $(y_1, \ldots., y_n)$, and this immediately shows that $(y_1, \ldots., y_n)$ are integrals of a linear differential equation with uniform coefficients.

But there are other cases in which Λ functions are connected with linear differential equations. Let S denote any substitution with uniform coefficients, and let us put

$$y_i = S(\eta_i), \quad \eta_i = S^{-1}(y_i).$$

Let $\mathfrak{A}_k$ be the substitution which $\eta_1, \ldots., \eta_n$ suffer if x makes a circuit around $x = a_k$. Then

$$A_k = S\mathfrak{A}_k S^{-1}.$$

Now if all the substitutions $\mathfrak{A}_k$ have constant coefficients, $\eta_1, \ldots., \eta_n$ are a fundamental system of integrals of a linear differential equation, and $y_1, \ldots., y_n$ therefore verify a system of n linear differential equations of the nth order. If therefore all of the substitutions A_k can be expressed in the form

$$A_k = S\mathfrak{A}_k S^{-1},$$

where $\mathfrak{A}_k$ has constant coefficients, and we put

$$\eta_i = S^{-1}(y_i),$$

then $\eta_1, \ldots\ldots, \eta_n$ constitute a fundamental system of integrals for a linear differential equation of the n^{th} order.*

It may also happen that certain linear homogeneous combinations of $(y_1, \ldots., y_n)$ and of the derivatives of $y_1, \ldots., y_n$ up to a certain order, can be found such that they are cogredient with $y_1, \ldots., y_n$. If those combinations are repeated n times, a system of linear differential equations is obtained which $y_1, \ldots., y_n$ must verify.

§8.—*Case that all Substitutions are Transformed into the Canonical Form by the same Transformation.*

The results of §5 obviously prove the existence of Λ functions having only two branch-points by giving their complete expression, the branch-points being $x = a$ and $x = \infty$ in the formulæ. But instead of the latter, any other point may of course be taken.† The functions $z_1, \ldots., z_n$ which are canonical for $x = a$, are also canonical for $x = b$.

In all cases in which the functions $z_1, \ldots., z_n$ are canonical for all singular points $a_1, a_2, \ldots., a_m$ at the same time, we can construct our functions without difficulty. Let us confine ourselves to this special case, which will be useful by showing in a very obvious manner certain relations which in the more general cases remain more or less obscure.

Our case will be even more special than this. Let all characteristic equations have unequal, absolutely uniform roots. In addition to the m branch-points, $a_1, a_2, \ldots., a_m$ we will take $x = \infty$ as branch-point (in general) with its substitution A_∞ so chosen that the relation is verified:

$$A_1 A_2 \ldots. A_m A_\infty = 1.$$

The n roots of the characteristic equation belonging to a_i are to be denoted by $\omega_{i1}, \omega_{i2}, \ldots., \omega_{in}$. Then $y_1, \ldots., y_n$ will be homogeneous linear functions of

$$z_k = \prod_{i=1}^{m} E(\omega_{ik})\, \phi_k(x), \qquad (k = 1, 2, \ldots., n)$$

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[*] I have been informed that Halphen has noticed this, but I have not been able to find the reference in the books at my disposal.

[†] By a linear transformation.

with uniform coefficients. $\phi_k(x)$ are uniform functions. This shows that these functions as determined by their group still contain n arbitrary uniform functions.

Since each z function is a homogeneous linear function of $y_1, \ldots, y_n$ with uniform coefficients, $z_1, \ldots, z_n$ must be uniform everywhere except in the vicinity of $a_1, a_2, \ldots, a_m, \infty$. Now let b be a point which coincides with neither of these points, and let $x = b$ be a zero or infinity of some of the ω_{ik} functions. For briefness we will omit the subscript k, thus

$$z = \prod_{i=1}^{m} E(\omega_i)\,\phi(x).$$

If ω_i has a_i for a zero or pole (counting the latter as a negative zero), we have

$$E(\omega_i) = e^{\frac{1}{2\pi i}\log(x-a_i)\left[\log\omega i + \frac{\mu i}{2}\log(x-a_i) - \mu\pi i\right]},$$

and this holds also if a_i is no zero of ω_i, if then we put $\omega_i' = \omega_i$, $\mu_i = 0$.

Let $x = b$ be a zero of ω_i of multiplicity ν_i. Then

$$\omega_i' = (x - b)^{\nu_i}\omega_i'',$$

where ω_i'' is neither zero nor infinite for $x = b$. But then in general b will become a branch-point for z. z will be uniform in the vicinity of $x = b$ only if the coefficient of $\log(x - b)$ in the exponential vanishes, i. e. if

$$\sum_{i=1}^{m} \nu_i \log(x - a_i) = 0,$$

which is only possible if $\nu_i = 0$, i. e. the points a_i and the multipliers ω_i cannot both be taken arbitrarily. If the singular points a_i are given, the multipliers must be so chosen that they have neither zeros nor poles which are different from these points. But, as is seen in the same way, a_k also can neither be a zero nor a pole of ω_i except for $i = k$.

This suffices to show that the zeros and poles of the roots of the characteristic equations merit especial attention. The results of this paragraph can partly be generalized, but it is not our intention to push this investigation any further at present.

§9.—*Generalized Hypergeometric Functions.*

It can be easily shown that the theorems of Gauss for the hypergeometric series $F(\alpha, \beta, \gamma, x)$, such as the "relationes inter functiones contigues," the development into a continued fraction, the criteria of convergence, the relations between functions having as fourth element x and $1-x$* and the representation by definite integrals, have the character of identities and do not essentially rest upon the assumption that α, β, γ are independent of x. If then α, β, γ are taken as uniform functions of x, which is the generalization which we have in mind, these theorems will remain true. The substitutions which two functions, for instance those defined in the vicinity of $x=0$ by

$$\begin{cases} y_1 = F(\alpha, \beta, \gamma, x), \\ y_2 = \alpha^{1-\gamma} F(a+1-\gamma, \ \beta+1-\gamma, \ 2-\gamma, \ x), \end{cases}$$

undergo when x makes circuits around $x=0$, $x=1$ and $x=\infty$ will then have the same form as when α, β, γ are constants, but their coefficients are now functions of x. Of course y_1 and y_2 are no longer solutions of a linear differential equation in general.

The validity of the relations between the above functions y_1 and y_2 and hypergeometric series, whose fourth element is $1-x$, is especially important from our point of view. The proof usually adopted which depends upon the differential equation is of course unsuitable for the general case. But we can immediately apply that method of proof which starts from the definition by a definite integral.

The relation

$$F(\alpha, \beta, \gamma, x) = \frac{\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-xu)^{-\alpha} du}{\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1} du}, \tag{18}$$

in which the denominator equals $B(\beta, \gamma-\beta)$, remains true if α, β, γ are functions of x, provided that

$$\Re(\beta) > 0, \quad \Re(\gamma-\beta) > 0, \quad \Re(\alpha) < 1, \tag{19}$$

i. e. for such values of x for which these inequalities hold. But if these inequalities are not fulfilled we will have to consider the double-loop integrals which have a definite sense, whatever values α, β, γ may have.

We could essentially follow the course of reasoning given by Klein in his "Vorlesung, Ueber die hypergeometrische Funktion," Göttingen, 1894. Only some obvious points need closer consideration if we wish to determine the function theoretic character of our functions in that way directly.

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* Called by Fuchs "Uebergangssubstitutionen."

It suffices to give the result. The double-loop integrals

$$\left.\begin{aligned} y_1 &= \int_{(0,\,x)} u^{\alpha-\gamma}(u-1)^{\gamma-\beta-1}(x-u)^{-\alpha}du\,, \\ y_2 &= \int_{(1,\,x)} u^{\alpha-\gamma}(u-1)^{\gamma-\beta-1}(x-u)^{-\alpha}du\,; \end{aligned}\right\} \tag{20}$$

the integration being on a double loop around 0 *and* x, *and around* 1 *and* x *respectively, form the elements of a* Λ *function with the branch-points* 0, 1, ∞. *The substitutions belonging to* 0 *and* 1 *are*

$$A_0 = \begin{pmatrix} e^{-2\pi i\gamma}, & 0 \\ -(1-e^{2\pi i(\gamma-\beta)}), & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1, & -e^{-2\pi i\alpha}(1-e^{2\pi i(\alpha-\gamma)}) \\ 0, & e^{2\pi i(\gamma-\beta-\alpha)} \end{pmatrix}, \tag{21}$$

while, of course,

$$A_\infty = A_0^{-1}A_1^{-1}.^*$$

The points $x = 0,\ 1,\ \infty$ are not the only singular points of the Λ function however. But these other singularities are only infinities in whose vicinity the function is uniform. They can be easily found from the representation in terms of the hypergeometric series.

The Λ function (20) will not in general verify an algebraical differential equation with rational coefficients. For if we put $\gamma-\beta-1=0$, y_1 will essentially become equal to an Eulerian integral of the first kind $B(\alpha-\gamma+1,\ -\alpha+1)$. Putting further $\gamma=1$, and therefore $\beta=0$,

$$y = B(\alpha-\gamma+1,\ -\alpha+1) = B(\alpha,\ -\alpha+1) = \frac{\Gamma(\alpha)\,\Gamma(1-\alpha)}{\Gamma(1)} = \frac{\pi}{\sin\pi\alpha}.$$

This verifies the equation

$$\left(\frac{dy}{d\alpha}\right)^2 = y^4\left(1-\frac{y^2}{\pi^2}\right),$$

whence

$$\left(\frac{dy}{dx}\right)^2 = y^4\left(1-\frac{y^2}{\pi^2}\right)\left(\frac{d\alpha}{dx}\right)^2.$$

Therefore $y,\ \dfrac{dy}{dx},\ \ldots\ldots,\ \dfrac{d^ny}{dx^n}$ can be expressed as algebraical functions of $\sin\pi\alpha,\ \dfrac{d\alpha}{dx},\ \ldots\ldots,\ \dfrac{d^n\alpha}{dx^n}$. Now if there were an algebraic relation between

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* For the proof when α, β, γ are constants, see Schleisinger, Hand. d. Lin. Diffgl. vol. II, pp. 460, 463.

$y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}$ with coefficients algebraic in x, there would be a similar relation between $\sin \pi\alpha, \frac{d\alpha}{dx}, \ldots, \frac{d^n\alpha}{dx^n}$. Computing $\sin \pi\alpha$ from this relation, differentiating with respect to x, and eliminating $\sin \pi\alpha$, an algebraical relation between $\frac{d\alpha}{dx}, \ldots, \frac{d^{n+1}\alpha}{dx^{n+1}}$ results. But we can choose α as uniform function of x, so that this is impossible. For instance, if we put

$$\alpha = \Gamma(x),$$

a function which, according to Mr. Hölder,* verifies no algebraical differential equation with rational coefficients, y will have the same property. In general, then, if α, β, γ are any uniform functions, the Λ function will not verify a differential equation with rational coefficients. It is an entirely different question, however, whether it may verify such an equation with uniform transcendental functions.

We will prove one general theorem about such differential equations in the next paragraph.

The Λ functions found in this paragraph can be easily generalized. The essential property, which must be retained in this generalization, is the representation by definite integrals.

§10.—*Differential Equations.*

Suppose that $(y_1, \ldots, y_n)$ is a Λ function, and that at least one of the characteristic equations belonging to some one branch-point, has absolutely uniform and distinct roots. Then there exist n functions

$$z_i = \lambda_{i1} y_1 + \lambda_{i2} y_2 + \ldots + \lambda_{in} y_n, \qquad (22)$$
$$(i = 1, 2, \ldots, n)$$

where λ_{ik} are uniform functions which, for a circuit around that branch-point a, undergo the substitution

$$\bar{z}_i = \omega_i z_i. \qquad (i = 1, 2, \ldots, n) \qquad (23)$$

Now suppose that $(y_1, \ldots, y_n)$ verify a system of algebraical differential equations with uniform coefficients,

$$D_k(y_1, y_2, \ldots, y_n) = 0. \qquad (k = 1, 2, \ldots, n) \qquad (24)$$

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* Hölder, Math. Ann., vol. 28, or Moore, Math. Ann., Bd. 48.

Then substituting the values of y_i found from (22), $z_1, \dots\dots, z_n$ will also verify such a system with uniform coefficients

$$\Delta_k(z_1, z_2, \dots\dots, z_n) = 0. \qquad (k = 1, 2, \dots\dots, n) \qquad (25)$$

But from (25) a system of n differential equations can be deduced, each of which contains only one unknown function z_k. These equations will appear in the form

$$\frac{d^{n_k} z_k}{dx^{n_k}} = F_k\left(x, z_k, \frac{dz_k}{dx}, \dots\dots, \frac{d^{n_k-1} z_k}{dx^{n_k-1}}\right),$$

where F_k is an algebraic function of $z_k, \dots\dots, \frac{dx^{n_k-1} z_k}{dx^{n_k-1}}$ and of the coefficients of (25). These equations can therefore be reduced to the form

$$E_k(z_k) = 0, \qquad (k = 1, 2, \dots\dots, n) \qquad (26)$$

where E_k is a rational integral function of $z_k, \dots\dots, \frac{d^{n_k} z_k}{dx^{n_k}}$ with coefficients which are uniform functions of x.

Let
$$E(z) = 0 \qquad (27)$$

be any one of these equations. If x describes a circuit around $x = a$, z changes into ωz, where ω is a uniform function, while the coefficients of E remain unaltered. We must therefore also have

$$E(\omega z) = 0, \qquad (28)$$

i. e. ωz must also be a solution of (27).

Let the equation be of the n^{th} order and of the m^{th} degree. We can write for the left member of (27),

$$E(y) = E_m(y) + E_{m-1}(y) + \dots\dots + E_1(y) + E_0(y), \qquad (29)$$

denoting by E_k a homogeneous polynomial of degree k,

$$E_k(y) = \Sigma p_{i_1 i_2 \dots\dots i_k} y^{(i_1)} y^{(i_2)} \dots\dots y^{(i_k)}, \qquad (30)$$
$$(i_1, i_2, \dots\dots, i_k = 0, 1, 2, \dots\dots, n)$$

where $p_{i_1 i_2 \dots\dots i_k}$ are uniform functions.[*]

In such an assemblage of terms we will call those terms the *highest* for which the sum of the indices

$$i_1 + i_2 + \dots\dots + i_k$$

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[*] $y^{(i_\lambda}$ **denotes** $\frac{d^{(i_k)} y}{dx^{i_\lambda}}$ **as usual.**

has the greatest value. We always have

$$i_1 + i_2 + \dots + i_k \leqq kn. \qquad (k = 1, 2, \dots, m)$$

Now (29) vanishes for $y = z$ or for $y = \omega z$. Putting $y = \omega z$, we have

$$y^{(i)} = \omega z^{(i)} + \binom{i}{1} \omega' z^{(i-1)} + \dots,$$

which shows that the highest terms in $E_k(\omega z)$ differ from those in $E_k(z)$ only by the factor ω^k. If we divide $E(\omega z)$ by ω^m, which is of course different from zero, the highest terms of $E(z)$ and $\frac{1}{\omega^m} E(\omega z)$ will be identical. If all of the other terms of these two expressions are not identical we will have another differential equation for z,

$$E(z) - \frac{1}{\omega^m} E(\omega z) = 0,$$

in which the highest terms of the m^{th} degree are at least by one lower than in (27). We will say that this second equation has a lower *type* than the first. If, then, z does not verify a differential equation of lower type than (27), a supposition which we can make without loss of generality, the corresponding coefficients of $E(z)$ and $\frac{1}{\omega^m} E(\omega z)$ must be equal.

Now suppose that $p_{k_1 k_2 \dots k_{m-1}} z^{(k_1)} z^{(k_2)} \dots z^{(k_{m-1})}$ is the highest term in $E_{m-1}(z)$. The corresponding term in $E_{m-1}(\omega z) \frac{1}{\omega^m}$ is $p_{k_1 \dots k_{m-1}} z^{(k_1)} z^{(k_2)} \dots z^{(k_{m-1})} \omega^{m-1} \frac{1}{\omega^m}$.

Therefore

$$p_{k_1 k_2 \dots k_{m-1}} = \frac{1}{\omega} p_{k_1 k_2 \dots k_{m-1}},$$

which would give $\omega = 1$ if $p_{k_1 k_2 \dots k_{m-1}}$ did not vanish. But that would make z uniform in the vicinity $x = a$, which is contrary to our supposition. Therefore

$$p_{k_1 k_2 \dots k_{m-1}} = 0.$$

And since this is the highest term of $E_{m-1}(z)$,

$$E_{m-1}(z) = E_{m-2}(z) = \dots = E_0(z) = 0,$$

i. e. the differential equation reduces to

$$E_m(z) = 0.$$

It is homogeneous. But according to (22) it is at once seen that *if the characteristic equation belonging to at least one branch-point has absolutely uniform and distinct roots, the system of differential equations with uniform coefficients, if any exists, which the elements of this Λ function verify, consists of homogeneous equations.*

Let us give a simple example. In the generalized hypergeometric series,

$$y = 1 + \frac{\alpha . \beta}{1 . \gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 . 2 . \gamma(\gamma+1)} x^2 + \ldots . ,$$

let $\beta = \gamma$, and let α be any uniform function. Then

$$y = 1 + \frac{\alpha}{1!} x + \frac{\alpha(\alpha+1)}{2!} x^2 + \ldots . = (1-x)^{-\alpha} = e^{-\alpha \log [1-x]} .$$

Therefore

$$\frac{1}{y} \frac{dy}{dx} = + \frac{\alpha}{1-x} - \frac{d\alpha}{dx} \log(1-x),$$

$$\frac{1}{y} \frac{d^2y}{dx^2} - \frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 = + \frac{\alpha}{(1-x)^2} + 2 \frac{d\alpha}{dx} \frac{1}{1-x} - \frac{d^2\alpha}{dx^2} \log(1-x).$$

Whence

$$\frac{d^2\alpha}{dx^2} \frac{1}{y} \frac{dy}{dx} - \frac{d\alpha}{dx}\left[\frac{1}{y} \frac{d^2y}{dx^2} - \frac{1}{y^2}\left(\frac{dy}{dx}\right)^2\right]$$
$$= \left[\alpha \frac{d^2\alpha}{dx^2} - 2\left(\frac{d\alpha}{dx}\right)^2\right] \frac{1}{1-x} - \alpha \frac{d^2\alpha}{dx^2} \frac{1}{(1-x)^2},$$

or

$$y \frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 - \frac{\frac{d^2\alpha}{dx^2}}{\frac{d\alpha}{dx}} y \frac{dy}{dx} = y^2 \left[\alpha \frac{\frac{d^2\alpha}{dx^2}}{\frac{d\alpha}{dx}} \frac{1}{(1-x)^2} - \left(\alpha \frac{\frac{d^2\alpha}{dx^2}}{\frac{d\alpha}{dx}} - 2 \frac{d\alpha}{dx}\right) \frac{1}{1-x}\right],$$

a homogeneous quadratic equation for y. If α is a rational function, the coefficients are also rational. The equation is a non-homogeneous linear equation for $\log y$. The most general homogeneous quadratic equation of the second order which can be satisfied by Λ functions is such an equation, linear in $\log y$.

§11.—*Conclusion.*

Confining ourselves to the behavior of Λ functions in the vicinity of any one branch-point, we found no difficulty in deducing general results. But we could only study the functions for all values of x by limiting ourselves to such special cases in which we could give analytical expressions defining them. In the case of constant coefficients of the substitutions not very much more has been done, the group is supposed to be given. A direct existence-theorem as required by Riemann's principles has only been given by Klein for a linear differential equation of the second order, the roots of whose determinating fundamental equation are real. We have not been able to find any proof of this kind, but have been satisfied in verifying the existence of such functions expressible as definite integrals.

There are some lines of investigation which seem to be promising. First, we can investigate the Λ functions expressible by definite integrals. Secondly, starting from the conception of cogrediency, we can attempt to formally find systems of combinations of $(y_1, \ldots, y_n)$ and their derivatives, which are cogredient to $(y_1, \ldots, y_n)$ and thus find differential equations which the functions belonging to a certain group must verify, if they exist. Finally, these equations will also be found by continuing the formal process used in §10. The conditions that the differential equations thus found are integrated by Λ functions can then be investigated, as well as the functions themselves. The inversion problems associated with such functions will be of especial interest and importance.

CHICAGO, *Feb.* 27, 1898.